Random matrix theory for mixed regular-chaotic dynamics in the super-extensive regime

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Abstract

We apply Tsallis's q-indexed nonextensive entropy to formulate a random matrix theory (**RMT**), which may be suitable for systems with mixed regular-chaotic dynamics. We consider the super-extensive regime of $q < 1$. We obtain analytical expressions for the level-spacing distributions, which are strictly valid for 2×2 random-matrix ensembles, as usually done in the standard **RMT**. We compare the results with spacing distributions, numerically calculated for random matrix ensembles describing a harmonic oscillator perturbed by Gaussian orthogonal and unitary ensembles.

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1 INTRODUCTION

The past decade has witnessed a growing interest in Tsallis' non-extensive generalization of statistical mechanics [1, 2]. The formalism has been successfully applied to a wide class of phenomena; for a review see, e.g. [2, 3]. The standard statistical mechanics is based on the Shannon entropy measure $S = -\int dx f(x) \ln f(x)$ (we use Boltzmann's constant $k_B = 1$), where $f(x)$ is a probability density function. This entropy is extensive. For a composite system $A+B$, constituted of two independent subsystems A and B such that the probability $p(A+B) = p(A)p(B)$, the entropy of the total system $S(A+B) = S(A)+S(B)$. Tsallis proposed a non-extensive generalization: $S_q = (1 - \int dx\, [f(x)]^q)/(q-1)$. The standard statistical mechanics is recovered when the entropic index $q = 1$. The value of q characterizes the degree of extensivity of the system. The entropy of the composite system $A + B$, the Tsallis' measure verifies

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B), \tag{1}$$

from which the denunciation non-extensive comes. Therefore, $S_q(A+B) < S_q(A) + S_q(B)$ if $q > 1$. This case is called sub-extensive. If $q < 1$, the system is in the super-extensive regime. The relation between the parameter q and the underlying microscopic dynamics is not fully understood yet.

Non-extensive generalization to the random matrix theory (**RMT**) [4] has recently received considerable attention [5, 6]. The conventional **RMT** is a statistical theory of random matrices $\mathbf{H}$ whose entries fluctuate as independent Gaussian random variables. The matrix-element distribution is obtained by extremizing Shannon's entropy subject to the constraint of normalization and existence of the expectation value of $\mathrm{Tr}(\mathbf{H}^{+}\mathbf{H})$ [7]. Non-extensive generalizations of **RMT**, on the other hand, extremize Tsallis' non-extensive entropy, rather than Shannon's. The first attempt in this direction is probably due to Evans and Michael [8]. Toscano et al. [9] constructed non-Gaussian ensemble by minimizing Tsallis' entropy and obtained expressions for the level densities and spacing distributions. A slightly different application of non-extensive statistical mechanics to **RMT** is due to Nobre and Souza [10]. Recently, Bertuola et al. [11] generated a new family of ensembles that unifies some important applications of **RMT**. In the range $-\infty < q < 1$, this family was found to be a restricted trace ensemble that interpolates between the bounded trace ensemble (Chapter 19 of [4]) at $q = -\infty$ and the Wigner Gaussian ensemble at $q = 1$. The non-extensive formalism was applied in Ref. [12] to ensembles of 2×2 matrices. Analytical expressions for the level-spacing distributions of mixed systems belonging to the orthogonal-symmetry universality class are obtained in [12]. The calculation of the spacing distribution showed different behavior depending on whether q is above or below 1. It is found that the sub-extensive regime of $q > 1$ [13] correspond to the evolution of a mixed system from chaos modelled by the standard **RMT** to order described by the Poisson statistics. On the other hand, the spectrum behaves in a different way in the super-extensive regime, where $q < 1$. Starting with a value of $q = 1$, where the nearest-neighbor spacing (**NNS**) distribution is well approximated by the Wigner surmise (see below), and decreasing q the distribution becomes narrower, and more sharply peaked at spacing $s = 1$. It develops towards the picked-fence type, such as the one obtained by Berry and Tabor [14] for the two-dimensional harmonic oscillator with non-commensurate frequencies.

The present paper considers the departure from chaos in the super-extensive regime. Section 2 reviews the non-extensive formulation of **RMT**. It contains the derivation of the

level spacing distributions for super-extensive systems with and without time reversible symmetry. Section 3 is devoted to a numerical experiment in which a two-dimensional harmonic oscillator is perturbed by a random-matrix ensemble. It is shown that the super-extensive **RMT** provides a reasonable description of the final stage of the stochastic transition. The conclusion of this work is given in Section 4.

2 Non-extensive RMT

RMT replaces the Hamiltonian of the system by an ensemble of Hamiltonians whose matrix elements are independent random variables. Dyson [15] showed that there are three generic ensembles of random matrices, defined in terms of the symmetry properties of the Hamiltonian. Time-reversal-invariant quantum system are represented by a Gaussian orthogonal ensemble (**GOE**) of random matrices when the system has rotational symmetry and by a Gaussian symplectic ensemble (**GSE**) otherwise. Chaotic systems without time reversal invariance are represented by the Gaussian unitary ensemble (**GUE**). The dimension β of the underlying parameter space is used to label these three ensembles: for **GOE**, **GUE** and **GSE**, β takes the values 1, 2 and 4, respectively. Balian [7] derived the weight functions $P_\beta(\mathbf{H})$ for the three Gaussian ensembles from the maximum entropy principle postulating the existence of a second moment of the Hamiltonian. He applied the conventional Shannon definition for the entropy to ensembles of random matrices as $S = -\int \mathrm{d}\mathbf{H} P_\beta(\mathbf{H}) \ln P_\beta(\mathbf{H})$ and maximized it under the constraints of normalization and fixed mean value of $\mathrm{Tr}(\mathbf{H}^{+}\mathbf{H})$ and obtained

$$P_\beta^G(\eta, \mathbf{H}) = Z_\beta^{-1} \exp\left[-\eta \mathrm{Tr}\left(\mathbf{H}^{+}\mathbf{H}\right)\right], \tag{2}$$

which is a Gaussian distribution with inverse variance $1/2\eta$. Here Z_β^{-1} is a normalization constant.

The non-extensive **RMT** applies the maximum entropy principle, with Tsallis' entropy, to obtain matrix-element distributions that are no more independent. The Tsallis entropy is defined for the joint matrix-element probability density $P_\beta(q, \mathbf{H})$ by

$$S_q\left[P_\beta(q, \mathbf{H})\right] = \left(1 - \int \mathrm{d}\mathbf{H}\left[P_\beta(q, \mathbf{H})\right]^q\right) \Big/ (q-1). \tag{3}$$

We shall refer to the corresponding ensembles as the Tsallis orthogonal ensemble (**TsOE**), the Tsallis Unitary ensemble (**TsUE**), and the Tsallis symplectic ensemble (**TsSE**). For $q \to 1$, S_q tends to Shannon's entropy, which yields the canonical Gaussian orthogonal, unitary or symplectic ensembles (**GOE**, **GUE**, **GSE**) [4, 7].

There are more than one formulation of non-extensive statistics which mainly differ in the definition of the averaging. Some of them are discussed in [16]. We apply the most recent formulation [17]. The probability distribution $P_\beta(q, \mathbf{H})$ is obtained by maximizing the entropy under two conditions,

$$\int \mathrm{d}\mathbf{H} P_\beta(q, \mathbf{H}) = 1, \tag{4}$$

$$\frac{\int \mathrm{d}\mathbf{H}\left[P_\beta(q, \mathbf{H})\right]^q \mathrm{Tr}\left(\mathbf{H}^{+}\mathbf{H}\right)}{\int \mathrm{d}\mathbf{H}\left[P_\beta(q, \mathbf{H})\right]^q} = \sigma_\beta^2, \tag{5}$$

where the constant σ_β is related to the variances of the matrix elements. The optimization of S_q with these constraints yields a power-law type for $P_\beta(q, H)$, which may be expressed as [12]

$$P_\beta(q, \mathbf{H}) = \widetilde{Z}_q^{-1} \left[1 - (1-q)\widetilde{\eta}_q \left\{ \mathrm{Tr}\left(\mathbf{H}^+\mathbf{H}\right) - \sigma_\beta^2 \right\}\right]_+^{1/(1-q)}, \tag{6}$$

where $\widetilde{\eta}_q = \eta / \int \mathrm{d}\mathbf{H}\,[P_\beta(q,\mathbf{H})]^q$, η is the lagrange multiplier associated with the constraint in Eq. (5), and $\widetilde{Z}_q = \int \mathrm{d}\mathbf{H} \left[1 - (1-q)\widetilde{\eta}_q \left\{ \mathrm{Tr}\left(\mathbf{H}^+\mathbf{H}\right) - \sigma_\beta^2 \right\}\right]_+^{1/(1-q)}$ is a normalization constant. We use in Eq. (6) the notation $[u]_+ = \max\{0, u\}$.

We now calculate the joint probability density for the eigenvalues of the Hamiltonian $\mathbf{H}$. With $\mathbf{H} = \mathbf{U^{-1}XU}$, where $\mathbf{U}$ is a unitary matrix. For this purpose, we introduce the elements of the diagonal matrix of eigenvalues $\mathbf{X} = \mathrm{diag}(x_1, \cdots, x_N)$ of the eigenvalues and the independent elements of $\mathbf{U}$ as new variables. Then the volume element (3) has the form

$$\mathrm{d}\mathbf{H} = \left|\Delta_N\left(\mathbf{X}\right)\right|^\beta \mathrm{d}\mathbf{X} d\mu(\mathbf{U}), \tag{7}$$

where $\Delta_N\left(\mathbf{X}\right) = \prod_{n>m}(x_n - x_m)$ is the Vandermonde determinant and $d\mu(\mathbf{U})$ the invariant Haar measure of the unitary group [4]. In terms of the new variables, $\mathrm{Tr}(\mathbf{H}^+\mathbf{H}) = \sum_{i=1}^N x_i^2$ so that the right-hand side of Eq. (7) is independent of the angular variables in $\mathbf{U}$. Integrating Eq. (6) over $\mu(\mathbf{U})$ yields the joint probability density of eigenvalues in the form

$$P_\beta(q, x_1, \cdots, x_N) = C_\beta(q) \left| \prod_{n>m} (x_n - x_m) \right|^\beta \left[1 - (1-q)\eta_q \sum_{i=1}^N x_i^2 \right]^{\frac{1}{1-q}}, \tag{8}$$

here C_β is a normalization constant.

Analytical expressions for the nearest-neighbor spacing (**NNS**) distribution can be obtained for the 2×2 matrix ensembles. For the Gaussian ensembles, this approach leads to the well-known Wigner surmises

$$P_\beta(s) = a_\beta s^\beta \exp\left(-b_\beta s^2\right), \tag{9}$$

where $(a_\beta, b_\beta) = (\frac{\pi}{2}, \frac{\pi}{4})$, $\left(\frac{32}{\pi^2}, \frac{4}{\pi}\right)$ and $\left(\frac{2^{18}}{3^6\pi^3}, \frac{64}{9\pi}\right)$ for **GOE**, **GUE**, and **GSE**, respectively. They present accurate approximation to the exact results for the case of $N \to \infty$.

We consider the non-extensive generalization of the Wigner surmises, hoping that the results present a reasonable approximation to the physically interesting cases of large N, as they do well in the case of Gaussian ensembles. We rewrite Eq. (8) for the case of $N = 2$, and introduce the new variable $s = |x_1 - x_2|$ and $X = (x_1 + x_2)/2$ to obtain

$$P_\beta(q, s, X) = C_\beta(q) s^\beta \left[1 - (1-q)\eta_q \left(\frac{1}{2}s^2 + 2X^2\right)\right]^{\frac{1}{1-q}}. \tag{10}$$

Now, we consider the case of $q < 1$. For this case, the distribution in Eq. (9) has to be complemented by the auxiliary condition that the quantity inside the square bracket has to be positive. Thus, we integrate over X from $-z$ to z, where $z = \sqrt{1 - \frac{1}{2}(1-q)\eta_q s^2}$. We obtain the following expression for **NNS** distribution in the super-extensive regime

$$P_\beta(q, s) = a_\beta(q) s^\beta \left[1 - b_\beta\left(q\right) s^2\right]^{\frac{1}{1-q} + \frac{1}{2}}, \tag{11}$$

where

$$a_\beta(q) = \frac{2\left[b_\beta(q)\right]^{(\beta+1)/2}\Gamma\left(2+\frac{\beta}{2}+\frac{1}{1-q}\right)}{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\frac{3}{2}+\frac{1}{1-q}\right)}, \tag{12}$$

and

$$b_\beta(q) = \left[\frac{\Gamma\left(\frac{\beta+2}{2}\right)\Gamma\left(2+\frac{\beta}{2}+\frac{1}{1-q}\right)}{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\frac{5}{2}+\frac{\beta}{2}+\frac{1}{1-q}\right)}\right]^2. \tag{13}$$

Thus, in the case of conserved time reversal symmetry when $\beta = 1$,

$$a_1(q) = \left(\frac{2}{1-q}+3\right)b_1(q), \tag{14}$$

$$b_1(q) = \frac{\pi}{4}\left[\frac{\Gamma\left(\frac{5}{2}+\frac{1}{1-q}\right)}{\Gamma\left(3+\frac{1}{1-q}\right)}\right]^2. \tag{15}$$

In the absence of time reversal symmetry, $\beta = 2$, and

$$a_2(q) = \frac{4\Gamma\left(3+\frac{1}{1-q}\right)}{\sqrt{\pi}\Gamma\left(\frac{3}{2}+\frac{1}{1-q}\right)}\left[b_2(q)\right]^{3/2}, \tag{16}$$

$$b_2(q) = \frac{4}{\pi}\left[\frac{\Gamma\left(3+\frac{1}{1-q}\right)}{\Gamma\left(\frac{7}{2}+\frac{1}{1-q}\right)}\right]^2. \tag{17}$$

3 Numerical experiment

It has been suggested in [12] that the super-extensive regime corresponds to the evolution of the chaotic system towards a regular regime like that of a two-dimensional irrational oscillator described by Berry and Tabor [14]. The energy eigenvalues for such a system is given by

$$E_i = \hbar\omega\left(n+\alpha m\right), \tag{18}$$

where ω is the larger frequency and α is the frequency ratio while i stands for the quantum-number pair m and n. Berry and Tabor calculated 10 000 eigenvalues and constructed the histograms of $P(s)$ for oscillators with $\alpha = 1/\sqrt{2}$, $1/\sqrt{5}$, and e^{-1}. These histograms are compared in Fig. 1 with the spacing distribution $P_1(0,s)$, calculated using Eq. (11).
The figure shows that the distribution $P_1(0,s)$ does not exactly fit the numerical histograms. This situation here is similar to the case of sub-extensive regime considered in [12, 13]. It was found in these papers that the non-extensive **RMT** cannot reproduce the Poisson distribution that described generic regular system for any value of $q > 1$. It can only describe the evolution of the shape spacing distribution until its peak reaches roughly half the distance from the peak position of the Wigner distribution to that of the Poisson.

We shall demonstrate that the super-extensive distribution describes the final stage of a transition from a regular harmonic dynamics to chaos in a similar way as the sub-extensive theory describes the final stage of the stochastic transition of generic regular systems [12, 13].

To show this, we shall now consider the following matrix Hamiltonian

$$\mathbf{H}(g) = (1-g)\mathbf{H}_{\mathrm{HO}} + g\mathbf{H}_{\mathbf{GOE}}, \tag{19}$$

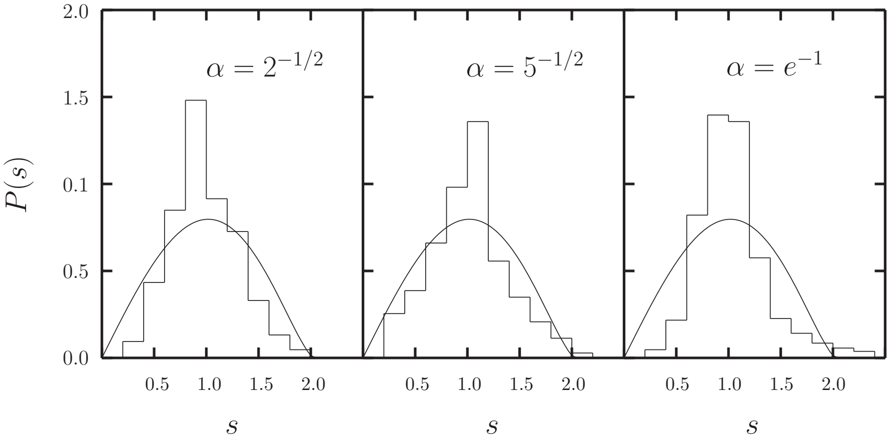

Figure 1: Comparison between **NNS** distributions for two-dimensional harmonic oscillators with different frequency ratios α obtained by Berry and Tabor [14], and the **NNS** distribution of the superstatistical ensemble with $q = 0$.

where $\mathbf{H}_{\mathrm{HO}} = \mathrm{diag}(E_i)$ is a diagonal matrix whose diagonal elements are given by Eq. (18) while $\mathbf{H}_{\mathbf{GOE}}$ is a random matrix of equal dimension with entries drawn from a **GOE**. By varying the parameter g from 0 to 1, the Hamiltonian $\mathbf{H}(g)$ describes a transition from a regular harmonic-oscillator type regime to chaotic dynamics.

We have considered ensembles of 20 matrices of dimension 200 $\times$ 200 representing the Hamiltonian in Eq. (19). We numerically diagonalized these matrices and calculated the **NNS** distributions for different values of the parameter g. We compared the resulting spacing distributions to Eq. (11) for $\beta = 1$ and evaluated the values of the entropic index parameter q that best fit these distributions. The results are shown in Fig. 2. We see that for $g \simeq 0.02$ we approach the **GOE** distribution. Fig. 3 shows the variation of the values of the entropic index q that best fit these distributions with the strength of the **GOE** perturbation g stating from $g = 0$ where $q = 0$ and the spacing distributions take the form of picked-fence type up to values of g where q approaches 1 and the spacing distributions become that of **GOE**.

We have considered perturbing the two-dimensional oscillator with a **GUE**, where the Hamiltonian takes the form

$$\mathbf{H}(g) = (1 - g)\mathbf{H}_{\mathrm{HO}} + g\mathbf{H}_{\mathbf{GUE}}, \tag{20}$$

Fig. 4 shows the **NNS** distributions of the two-dimensional oscillator perturbed by a **GUE** of strength g. We compare the resulting spacing distributions to Eq. (11) for $\beta = 2$ and evaluated the values of the entropic index q that best fit these distributions. Fig. 5 shows the variation of the entropic index q with the **GUE** perturbation strength g.

From Fig. 2 and Fig. 4, we see that Eq. (11) does not exactly fit the initial stage of the transition from the regular two-dimensional oscillator dynamics but fits the final stage of the transition into both **GOE** and **GUE**. From Fig. 3 and Fig. 5, we see that the **GUE** limit is approached faster than the **GOE** limit.

4 Conclusion

In summary, the **NNS** distribution obtained from applying Tsallis statistics to **RMT** maps two routes for the transition from chaos to order. One leads towards the integrability described by the Poisson statistics by increasing $q > 1$, but ends at the edge of chaos. The other is followed by decreasing q form 1 to 0. It leads to a picket-fence type spectrum, such as the one obtained by Berry and Tabor [14] for the two-dimensional harmonic oscillator with non-commensurate frequencies.

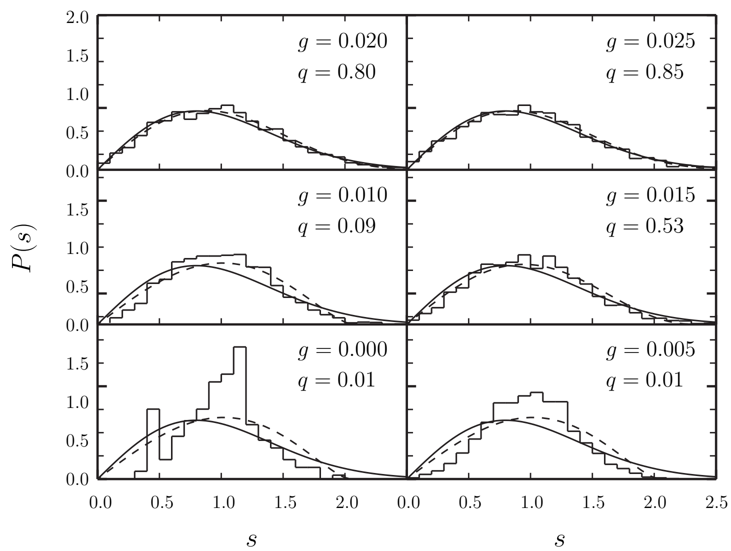

Figure 2: **NNS** distributions for the Hamiltonian matrix representing a two dimensional harmonic oscillator perturbed by a **GOE** Eq. (19) for different values of the parameter g. Dashed curves represent fitting the histograms to Eq. (11) for $\beta = 1$. The values of the entropic index q that best fit these distributions are given. The solid curves are pure **GOE**.

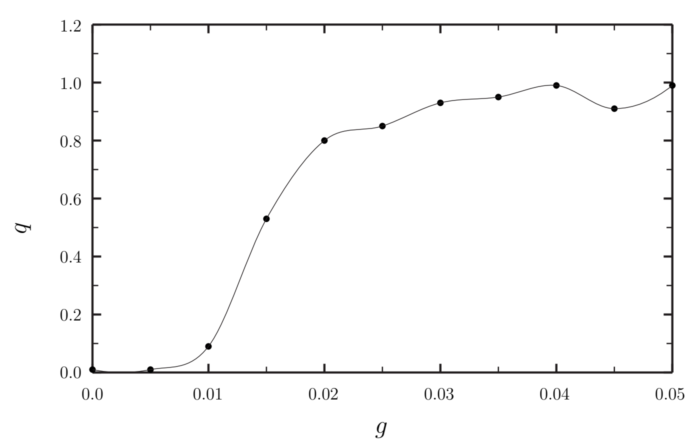

Figure 3: Variation the entropic index q of Eq.(11) for $\beta = 1$ that best fits the **NNS** distributions of the Hamiltonian in Eq. (19) with the parameter g.

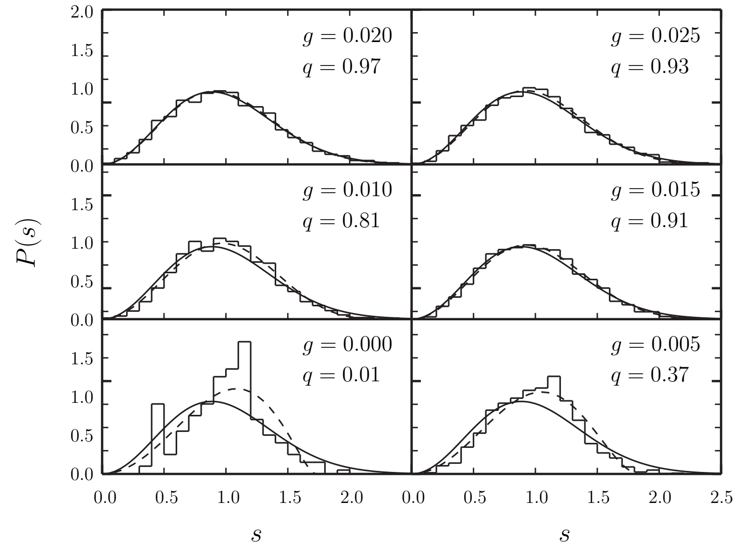

Figure 4: Same as Fig. 2, but for $\beta = 2$ and **GUE** perturbation.

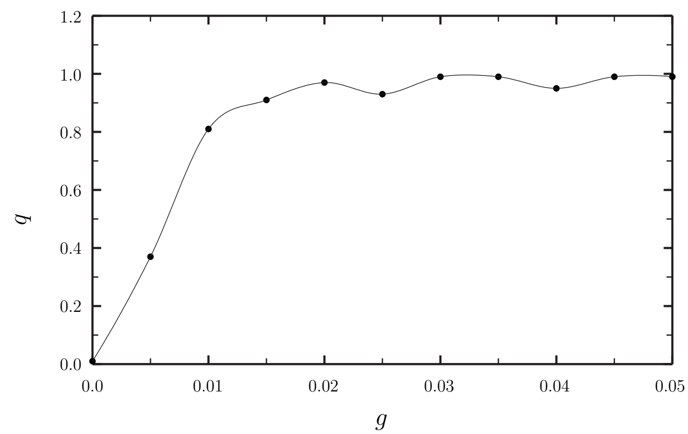

Figure 5: Same as Fig. 3, but for $\beta = 2$ and **GUE** perturbation.

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